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Determination of Gaussian Integer Zeroes of $F(x,z) = 2x^4 - z^3$

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Abstract

In this paper the zeroes of the polynomial $F(x,z) = 2x^4 - z^3$ in Gaussian integers $\mathbb{Z}[i]$ are determined, a problem equivalent to finding the solutions of the Diophatine equation $x^4 + y^4 = z^3$ in $\mathbb{Z}[i]$, with a focus on the case $x = y$. We start by using an analytical method that examines the real and imaginary parts of the equation $F(x,z) = 0$. This analysis sheds light on the general algebraic behavior of the polynomial $F(x,z)$ itself and its zeroes. This in turn allows us a deeper understanding of the different cases and conditions that give rise to trivial and non-trivial solutions to $F(x,z) = 0$, and those that lead to inconsistencies. This paper concludes with a general formulation of the solutions to $F(x,z) = 0$ in Gaussian integers. Results obtained in this work show the existence of infinitely many non-trivial zeroes for $F(x,z) = 2x^4 - z^3$ under the general form $x = (1+i)\eta^3$ and $c = -2\eta^4$ for $\eta \in \mathbb{Z}[i]$.

Keywords: Gaussian integer; Diophantine equation; prime power decomposition.

1 Introduction

Diophantine equations have been studied since ancient times. This activity has been mainly concentrated in the quest to find integer solutions to polynomial expressions over the ring of integers, when they exist. Since the time of Fermat, research of this type has progressed. For example, integer solutions to the well-known equation in Fermat's Last Theorem $x^n + y^n = z^n$ were finally proved to be nonexistent for $n \geq 3$ in 1995 by Andrew Wiles. Early this century Dieulefait [5] proved that, with p a prime, $p \not\equiv -1 \pmod 8$ and $p > 13$, the equation $x^4 + y^4 = z^p$ has no solutions x, y, z with $\gcd(x, y) = 1$ and $xy \neq 0$. In the same year Dieulefait [4] proved that the equation $x^4 + y^4 = qz^p$ does not have primitive solutions for $q = 73, 89, 113$ and $p > 13$. Poonen [9] studied the equation $x^n + y^n = z^2$ and showed that it has no non-trivial primitive solutions for $n \geq 4$. By assuming the Shimura-Taniyama conjecture he also showed that the equation $x^n + y^n = z^3$ has no non-trivial primitive solutions for $n \geq 3$. More recently, Bakar *et al.* [1] investigated the Diophantine equation $5^x + p^m n^y = z^2$, where $p > 5$ is a prime number and $y \in \{1, 2\}$, presenting the positive solutions under the forms $(x, m, n, y, z) = (2r, t, p^t k^2 \pm 2^k 5^r, 1, p^t k \pm 5^r)$ and $(x, m, n, y, z) = \left(2r, 2t, \frac{5^{2r-\alpha} - 5^\alpha}{2p^t}, 2, \frac{5^{2r-\alpha} + 5^\alpha}{2p^t}\right)$, for $k, r, t \in \mathbb{N}$ and $0 \leq \alpha < r$. Sihabudin *et al.*[10] discussed the system of simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - pz^2 = 1$, and found the solutions $(x, y, z, m) = (y_n^2 t \pm 1, y_n, z_n, y_n^2 t^2 \pm 2t)$ and $(x, y, z, m) = \left(\frac{y_n^2}{2} t \pm 1, y_n, z_n, \frac{y_n^2}{4} t^2 \pm t\right)$, for y_n odd or even, respectively, and $t \in \mathbb{N}$.

The activity of finding solutions to Diophantine equations then shifted to determining solutions in the larger ring of Gaussian integers. In the 90's, Cross [3] examined the Diophantine equation $x^4 + y^4 = z^4$. He employed cyclotomic integers in view of the particular case $n = 3$ of Fermat's Last Theorem, to show that no triplet (x, y, z) of non-zero Gaussian integers exists with $\gcd(x, y) = 1$ such that $\pm x^4 \pm y^4 = \pm z^4$. His proof uses a version of the infinite descent method.

Szabo [11] stated that for certain choices of the coefficients a, b, c, the solutions of the Diophantine equation $ax^4 + by^4 = cz^2$ in Gaussian integers satisfy $xy = 0$. Let p be a rational prime, $p \equiv 3 \pmod 8$, and let $d = p$ or $d = p^3$, then the equation $x^4 - dy^4 = z^2$ has only trivial solutions in $\mathbb{Z}[i]$. Similarly, when p is a rational prime, $p \equiv 3 \pmod 8$, then the equations $x^4 - y^4 = pz^2$ and $x^4 - p^2 y^4 = z^2$ have only trivial solutions in $\mathbb{Z}[i]$. Also, the author revealed that the equation $x^4 + 2y^4 = z^2$ has only trivial solutions in $\mathbb{Z}[i]$.

Najman [8] showed that the equation $x^4 - y^4 = iz^2$ has only trivial solutions in Gaussian integers. He also showed that the only non-trivial Gaussian solutions of the equation $x^4 + y^4 = iz^2$ satisfying $\gcd(x, y, z) = 1$ are $x, y \in \{\pm 1, \pm i\}$ and $z = \pm i(1 + i)$.

Izadi *et al.* [7] examined a class of fourth power Diophantine equations of the forms $x^4 + kx^2 y^2 + y^4 = z^2$ and $ax^4 + y^4 = cz^2$, in Gaussian integers, where a and b are prime integers. The results revealed that when $p \equiv 3 \pmod 8$, the Diophantine equations $y^4 - p^2 x^4 = \pm z^2$ and $y^4 + p^2 x^4 = \pm iz^2$ have only trivial solutions in $\mathbb{Z}[i]$. Likewise, for $p \equiv 5 \pmod 8$, the Diophantine equations $y^4 + p^2 x^4 = \pm z^2$ and $y^4 - p^2 x^4 = \pm iz^2$ have only trivial solutions in $\mathbb{Z}[i]$.

In 2002, Cohen [2] showed that the solution set in integers of the equation $x^4 + y^4 = z^3$ is non-empty whenever $\gcd(x, y, z) > 1$. In our quest to determine the form of Gaussian integers that satisfy this same equation, we begin by first restricting our investigation to the special case $x = y$, as presented in this paper. This quest thus transforms into one of finding Gaussian integer zeroes of $F(x, z) = 2x^4 - z^3$. We begin in the following section by identifying conditions under which the equation $F(x, z) = 0$ has roots in the ring of Gaussian integers, or otherwise.

2 Analysis and Discussion

This section presents an analytical study of the general behavior of the Diophantine equation $F(x,z) = 2x^4 - z^3 = 0$. The objective of this section is to present an analysis of the conditions that give rise to solutions, inconsistencies or trivial solutions to $F(x,z) = 0$.

2.1 Preamble

Suppose (a,c) is a zero of $F(x,z) = 2x^4 - z^3$, where $a, c \in \mathbb{Z}[i]$. Then, there exist integers r, s, g and h such that

$$a = r + si \qquad \text{and} \qquad c = g + hi. \tag{1}$$

Clearly, in order to determine the zeroes (a,c) of $F(x,z)$, it is necessary to find the integers r, s, g and h (that is, the real and imaginary parts of a and c, respectively) such that

$$2a^4 = c^3.$$

On expanding the expression we obtain

$$(2r^4 - 12r^2s^2 + 2s^4) + (8r^3s - 8rs^3)i = (g^3 - 3gh^2) + (3g^2h - h^3)i.$$

From (2.1), equating the real and the imaginary coefficients gives us

$$2r^4 - 12r^2s^2 + 2s^4 = g^3 - 3gh^2, \tag{2}$$

$$8r^3s - 8rs^3 = 3g^2h - h^3. \tag{3}$$

In the ensuing discussion, let

$$F_1(r,s,g,h) = 2r^4 - 12r^2s^2 + 2s^4 - g^3 + 3gh^2, \tag{4}$$

$$F_2(r,s,g,h) = 8r^3s - 8rs^3 - 3g^2h + h^3. \tag{5}$$

Then, from (2) and (3), respectively, we have $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$. It follows that finding common solutions to (2) and (3) is equivalent to looking for common solutions of (4) and (5).

We will now consider these two equations in our search for their common solutions. This will be carried out by looking at the relative sizes of their coefficients, as in the following section. Since the presence or non-existence of common solutions to the equations of a system depends on the consistency or inconsistency of the system, we will first determine the specific situations that display these characteristics. We start by identifying those that have trivial solutions.

2.2 Trivial Solutions to Equations $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$

The following lemma gives common trivial solutions under certain conditions on the coefficients of the equations $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$.

Lemma 2.1. *Let*

$$F_1(r,s,g,h) = 2r^4 - 12r^2s^2 + 2s^4 - g^3 + 3gh^2,$$
$$F_2(r,s,g,h) = 8r^3s - 8rs^3 - 3g^2h + h^3.$$

Then, the system $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$ will have common trivial solutions only when $g = h$.

Proof. Suppose $g = h$. From $F_1(r,s,g,h) = 0$, we have

$$r^4 - 6r^2s^2 + s^4 = -h^3, \tag{6}$$

while from $F_2(r,s,g,h) = 0$, we have

$$4r^3s - 4rs^3 = h^3. \tag{7}$$

Upon substituting (6) into (7), we will have

$$r^4 - 6r^2s^2 + s^4 = 4rs^3 - 4r^3s. \tag{8}$$

By solving (8) for r, we obtain the following two possibilities:

$$r = \sqrt{2}s - s \pm |s|\sqrt{-2\sqrt{2}+4} = s\left(\sqrt{2} - 1 \pm \sqrt{-2\sqrt{2}+4}\right),$$
$$r = -\sqrt{2}s - s \pm |s|\sqrt{2\sqrt{2}+4} = s\left(-\sqrt{2} - 1 \pm \sqrt{2\sqrt{2}+4}\right).$$

It is evident that for any non-zero integer values of r and s, these possibilities all lead to a contradiction. Thus, $r = 0$ and $s = 0$. By (7), $h = 0$ and it follows that $g = 0$. Therefore, under the condition $g = h$, the equations $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$ will only have the common trivial solution $(r,s,g,h) = (0,0,0,0)$. □

It follows from Lemma 2.1 that the system $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$ has no nontrivial solutions when $g = h$.

The following lemma identifies cases of inconsistency under certain conditions on the coefficients of the equations $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$.

Lemma 2.2. *Let*

$$F_1(r,s,g,h) = 2r^4 - 12r^2s^2 + 2s^4 - g^3 + 3gh^2,$$
$$F_2(r,s,g,h) = 8r^3s - 8rs^3 - 3g^2h + h^3.$$

Then, the system $F_1(r,s,g,h) = 0$ and $F_2(r,s,g,h) = 0$ will be inconsistent in the following cases:

(i) $r = 0$, $s \neq 0$, $h \neq 0$, $g = 0$;

(ii) $r = 0$, $s \neq 0$, $h \neq 0$, $g \neq 0$;

(iii) $r \neq 0$, $s = 0$, $h \neq 0$, $g = 0$;

(iv) $r \neq 0$, $s = 0$, $h \neq 0$, $g \neq 0$;

(v) $r \neq 0$, $s \neq 0$, $h \neq 0$, $g = 0$.

Proof. Assume (i) $r = 0$, $s \neq 0$, $h \neq 0$, $g = 0$. Under this case, inconsistency arises as follows. From $F_2(r, s, g, h) = 0$, we obtain $h^3 = 0$, or $h = 0$, which gives rise to a contradiction since earlier it is stated that $h \neq 0$.

Suppose (ii) $r = 0$, $s \neq 0$, $h \neq 0$, $g \neq 0$. From $F_2(r, s, g, h) = 0$, we obtain

$$\frac{h}{g} = \pm\sqrt{3},$$

which gives rise to a contradiction since h/g is rational while $\sqrt{3}$ is irrational.

Assume (iii) $r \neq 0$, $s = 0$, $h \neq 0$, $g = 0$. This case leads to a similar inconsistency as in (i) due to the symmetrical conditions on r and s, as well as identical conditions on g and h.

Assume (iv) $r \neq 0$, $s = 0$, $h \neq 0$, $g \neq 0$. This case leads to a similar inconsistency as in (ii) due to the symmetrical conditions on r and s, as well as identical conditions on g and h.

Suppose (v) $r \neq 0$, $s \neq 0$, $h \neq 0$, $g = 0$. From $F_1(r, s, g, h) = 0$, we obtain $r^4 - 6r^2s^2 + s^4 = 0$, or

$$\frac{r^2 - 3s^2}{2s^2} = \pm\sqrt{2},$$

which gives rise to inconsistency since $\frac{r^2-3s^2}{2s^2}$ is rational while $\sqrt{2}$ is irrational. □

The equations $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$ in Lemma 2.1 will yield common trivial solutions. Those having conditions (i) to (v) of Lemma 2.2 do not have common solutions due to their inconsistencies. In the following section we will look for non-trivial solutions in a consistent system $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$ by investigating their coefficients that do not have the conditions in both lemmas.

2.3 Non-trivial Common Solutions to $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$

We will show first in the following theorem that common solutions of $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$ can be generated from a known common solution of the two.

Theorem 2.1. *Let (t, u, v, w) be a quadruplet of common integer solutions of $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$. Then the following quadruplets are also common solutions to both equations: $(t, -u, v, -w)$, $(u, t, v, -w)$, $(u, -t, v, w)$, $(-t, u, v, -w)$, $(-t, -u, v, w)$, $(-u, t, v, w)$, $(-u, -t, v, -w)$.*

Proof. We are given that $F_1(t,u,v,w)=0$ and $F_2(t,u,v,w)=0$. Now,

$$\begin{aligned} F_1(t,-u,v,-w) &= 2t^4-12t^2(-u)^2+2(-u)^4-v^3+3v(-w)^2 \\ &= 2t^4-12t^2u^2+2u^4-v^3+3vw^2 \\ &= F_1(t,u,v,w)=0, \end{aligned}$$

and

$$\begin{aligned} F_2(t,-u,v,-w) &= 8t^3\left(-u\right)-8t\left(-u\right)^3-3v^2\left(-w\right)+\left(-w\right)^3 \\ &= -(8t^3u-8tu^3-3v^2w+w^3) \\ &= -F_2(t,u,v,w)=0. \end{aligned}$$

Thus, $(t,-u,v,-w)$ is a common solution of $F_1(r,s,g,h)=0$ and $F_2(r,s,g,h)=0$.

In a similar manner, it can be proved that the remaining quadruplets are common integer solutions of both equations. □

We now examine and study the coefficients of both equations and impose conditions on them that differ from those in Lemmas 2.1 and 2.2. First, we have the following assertion.

Lemma 2.3. *Let $r=0$ and $s\neq 0$, or $r\neq 0$ and $s=0$, both with $h=0$ and $g\neq 0$ in $F_1(r,s,g,h)=0$ and $F_2(r,s,g,h)=0$. Then, there exist integers $n>0$ such that $(s,g)=\left(\pm 4n^3,8n^4\right)$ or $(r,g)=\left(\pm 4n^3,8n^4\right)$, respectively.*

Proof. Suppose $r=0$, $s\neq 0$ with $h=0$, $g\neq 0$ in $F_1(r,s,g,h)=0$ and $F_2(r,s,g,h)=0$. From $F_1(r,s,g,h)=0$, we obtain $2s^4=g^3$. Since s and g are integers, from Theorem 1.2 in [6], the solutions to this equation are given by $(s,g)=\left(\pm 4n^3,8n^4\right)$, where $n>0$.

Similarly, suppose $r\neq 0$, $s=0$ with $h=0$, $g\neq 0$ in $F_1(r,s,g,h)=0$ and $F_2(r,s,g,h)=0$. From $F_1(r,s,g,h)=0$ we obtain $2r^4=g^3$. Since r and g are integers, from Theorem 1.2 in [6], the solutions to this equation are given by $(r,g)=\left(\pm 4n^3,8n^4\right)$, where $n>0$. In this case, it follows from (1) that $a=4n^3$ and $c=8n^4$, which leads back to the integer solutions obtained in [6]. □

The following lemma shows the existence of solutions to $F_1(r,s,g,h)=0$ and $F_2(r,s,g,h)=0$ under particular conditions that differ from those in Lemmas 2.1 and 2.2.

Lemma 2.4. *Let $r\neq 0$, $s\neq 0$, $h=0$, $g\neq 0$ in $F_1(r,s,g,h)=0$ and $F_2(r,s,g,h)=0$. Then there exist integers $n>0$ such that $(s,g)=(\pm 8n^3,-32n^4)$ and $(s,g)=(\pm n^3,-2n^4)$.*

Proof. From $F_2(r,s,g,h)=0$, we have

$$8r^3s-8rs^3=0.$$

It follows that

$$8rs(r^2-s^2)=0.$$

Since $r\neq 0$ and $s\neq 0$, we must have $r^2-s^2=0$, which implies $r=\pm s$, and hence $F_1(r,s,g,h)=0$ gives us

$$-8s^4=g^3. \tag{9}$$

We can see that $2|g^3$, which implies that g is even. Hence, there exists $q_j^{f_j}$, with $q_j = 2$ and $f_j \geq 1$, in the prime power decomposition of g. By rearranging, we let $q_1 = 2^{f_1}$. Also, it can be seen that g is negative. Thus, we let

$$g = -2^{f_1} \prod_{j=2}^{m} q_j^{f_j}, \text{ where } f_1 \geq 1 \text{ and } f_j \geq 0 \text{ for } j = 2, \dots, m. \tag{10}$$

Similarly, we let $p_1 = 2$ in the prime power decomposition of s. Then, we will have

$$s = \pm 2^{e_1} \prod_{i=2}^{l} p_i^{e_i}, \text{ where } e_1 \geq 0. \tag{11}$$

Upon substitution of (10) and (11) into (9), we obtain

$$2^{4e_1+3} \left(\prod_{i=2}^{l} p_i^{4e_i} \right) = 2^{3f_1} \left(\prod_{j=2}^{m} q_j^{3f_j} \right). \tag{12}$$

By uniqueness of prime power decomposition of s and g, we will have $2^{4e_1+3} = 2^{3f_1}$, which implies that $4e_1 + 3 = 3f_1$ or $3f_1 + (-4)e_1 = 3$. We can see that $e_0 = 3$ and $f_0 = 5$ satisfy this equation. Thus, all solutions for this equation are given by

$$e_1 = 3 - 3t_1 \qquad \text{and} \qquad f_1 = 5 - 4t_1, \tag{13}$$

where t_1 is an integer. Since e_1 is a positive integer, we have

$$3 - 3t_1 \geq 0 \qquad \text{or} \qquad t_1 \leq 1.$$

For $t_1 < 0$, there exists an integer $s_1 > 0$ such that $t_1 = -s_1$. Thus,

$$e_1 = 3 + 3s_1 \qquad \text{and} \qquad f_1 = 5 + 4s_1.$$

Also, from (12), by the uniqueness of prime power decomposition of s and g, we have $l = m$, and for each $i \in \{2, 3, \dots, l\}$, there exists $j \in \{2, 3, \dots, l\}$ such that

$$p_i = q_j \qquad \text{and} \qquad 4e_i = 3f_j. \tag{14}$$

From this, we can see that $4|3f_j$ and $3|4e_i$. Since $\gcd(4, 3) = 1$, there exist integers r_j and s_i such that $f_j = 4r_j$ and $e_i = 3s_i$. It follows from (14) that $s_i = r_j$. Then, the prime power decompositions of s and g become

$$s = \pm 2^{3+3s_1} \prod_{i=2}^{l} p_i^{3s_i} = \pm 8 \left(2^{s_1} \prod_{i=2}^{l} p_i^{s_i} \right)^3,$$

$$g = -2^{5+4s_1} \prod_{i=2}^{l} p_i^{4s_i} = -32 \left(2^{s_1} \prod_{i=2}^{l} p_i^{s_i} \right)^4.$$

Let $n = 2^{s_1} \prod_{i=2}^{l} p_i^{s_i}$. Thus, we obtain $n > 0$ with $s = \pm 8n^3$ and $g = -32n^4$ when $t_1 < 0$, as asserted.

For $t_1 = 0$, from (13), we obtain $e_1 = 3$ and $f_1 = 5$. Then, the prime power decompositions of s and g become $s = \pm 8 \left(\prod_{i=2}^{l} p_i^{s_i}\right)^3$ and $g = -32 \left(\prod_{i=2}^{l} p_i^{s_i}\right)^4$. Let $n = \prod_{i=2}^{l} p_i^{s_i}$ in this case. Thus, we obtain $s = \pm 8n^3$ and $g = -32n^4$ when $t_1 = 0$, as asserted.

For $t_1 = 1$, from (13), we obtain $e_1 = 0$ and $f_1 = 1$. Then, the prime power decompositions of s and g become $s = \pm \left(\prod_{i=2}^{l} p_i^{s_i}\right)^3$ and $g = -2 \left(\prod_{i=2}^{l} p_i^{s_i}\right)^4$. Let $n = \prod_{i=2}^{l} p_i^{s_i}$ in this case. Thus, we obtain $s = \pm n^3$ and $g = -2n^4$ when $t_1 = 1$, as asserted. $\square$

Lemma 2.5. *Let $r \neq 0$ and $r = \pm s$ with $g \neq h$ in $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$. Then, $h = 0$ and there exist integers $n > 0$ such that $(s, g) = (\pm n^3, -2n^4)$ and $(s, g) = (\pm 8n^3, -32n^4)$.*

Proof. Since $r = \pm s$, we have from $F_2(r, s, g, h) = 0$ that

$$3g^2h - h^3 = 0,$$

from which

$$h(3g^2 - h^2) = 0.$$

Hence, we have $h = 0$ or $3g^2 - h^2 = 0$. We claim that $h = 0$, for if $h \neq 0$, then $3g^2 - h^2 = 0$. This leads to $g/h = \pm 1/\sqrt{3}$, which is a contradiction, since $1/\sqrt{3}$ is irrational while g/h is rational.

Thus, the assertion $h = 0$ holds. Since $g \neq h$, it follows that $g \neq 0$. Moreover, since $r \neq 0$ and $r = \pm s$, clearly $s \neq 0$. Therefore, the values of (s, g) follow from Lemma 2.4. $\square$

2.4 Gaussian Integer Zeroes of $F(x, z) = 2x^4 - z^3$

The following theorem gives the solutions to the equation $F(x, z) = 2x^4 - z^3$ in Gaussian integers under the consistent system of equations $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$, as defined in (4) and (5), respectively.

Theorem 2.2. *Let $F(x, z) = 2x^4 - z^3$, and $a = r + si$ and $c = g + hi$ be Gaussian integers. Then,*

(*i*) *(a, c) is a trivial zero of $F(x, z)$ if $g = h$;*

(*ii*) *$(a, c) \in \{(\pm 4n^3, 8n^4), (\pm 4n^3 i, 8n^4)\}$ with $n > 0$ are zeroes of $F(x, z)$ if $r = 0$ and $s \neq 0$, or $r \neq 0$ and $s = 0$, both $h = 0$ and $g \neq 0$;*

(*iii*) *$(a, c) \in \{(\pm n^3 \pm n^3 i, -2n^4), (\pm n^3 \mp n^3 i, -2n^4), (\pm 8n^3 \pm 8n^3 i, -32n^4), (\pm 8n^3 \mp 8n^3 i, -32n^4)\}$ are zeroes of $F(x, z)$ if $r \neq 0$, $r = \pm s$ and $g \neq h$.*

Proof. Let $F_1(r, s, g, h)$ and $F_2(r, s, g, h)$ be defined as in (4) and (5), respectively.

(i) Let $g = h$. By Lemma 2.1, the equations $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$ will have only the common trivial solution $(r, s, g, h) = (0, 0, 0, 0)$. Hence, $(a, c) = (0, 0)$ is a trivial zero of $F(x, z)$.

(ii) Let $r = 0$ and $s \neq 0$, or $r \neq 0$ and $s = 0$, both with $h = 0$ and $g \neq 0$, such that $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$. By Lemma 2.3, there exists $n > 0$ such that

$$(s, g) = (\pm 4n^3, 8n^4) \qquad \text{or} \qquad (r, g) = (\pm 4n^3, 8n^4).$$

By Theorem 2.1,

$$(r, s, g, h) \in \{(0, \pm 4n^3, 8n^4, 0), (\pm 4n^3, 0, 8n^4, 0)\},$$

are common solutions to $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$. It follows that

$$(a, c) \in \{(\pm 4n^3, 8n^4), (\pm 4n^3 i, 8n^4)\},$$

are zeroes of $F(x, z)$.

(iii) Let $r \neq 0$, $r = \pm s$ and $g \neq h$, with $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$. By Lemma 2.5, there exists an integer $n > 0$ such that

$$(s, g) = (\pm 8n^3, -32n^4) \qquad \text{or} \qquad (s, g) = (\pm n^3, -2n^4).$$

Since $r = \pm s$ by hypothesis and $h = 0$ by Lemma 2.5, it follows from Theorem 2.1 that

$$\begin{aligned}(r, s, g, h) \in \{&(\pm n^3, \pm n^3, -2n^4, 0), (\pm n^3, \mp n^3, -2n^4, 0)\\ &(\pm 8n^3, \pm 8n^3, -32n^4, 0), (\pm 8n^3, \mp 8n^3, -32n^4, 0)\},\end{aligned}$$

are common solutions to $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$. Therefore,

$$\begin{aligned}(a, c) \in \{&(\pm n^3 \pm n^3 i, -2n^4), (\pm n^3 \mp n^3 i, -2n^4)\\ &(\pm 8n^3 \pm 8n^3 i, -32n^4), (\pm 8n^3 \mp 8n^3 i, -32n^4)\},\end{aligned}$$

are zeroes of $F(x, z)$. □

In the above discussion the zeroes of $F(x, z) = 2x^4 - z^3$ are sought by determining through elementary analytical methods the real and imaginary components of the Gaussian integer pairs $(a, c) = (r + si, g + hi)$ satisfying $F(a, c) = 0$, according to constraints on the sizes of the components. This allowed us to determine the existence or nonexistence of zeroes of $F(x, z)$ under each constraint. For those that have been determined to have non-trivial zeroes, we have obtained explicit expressions of the pairs (a, c) in terms of positive integers n, as asserted in Theorem 2.2.

The constraints imposed on the components are not exhaustive. Consequently, the results obtained do not represent all the zeroes of $F(x, z)$. They form a subset of the solutions to $F(x, z) = 0$. The following section discusses the general form of these solutions, as well as their real and imaginary components.

3 General Form of Solutions

The following result gives a general form for the zeroes of $F(x, z) = 2x^4 - z^3$ in Gaussian integers.

Theorem 3.1. *Let (a, c) be a zero of the polynomial $F(x, z) = 2x^4 - z^3$ in Gaussian Integers. Then*

$$a = (1 + i)\eta^3 \qquad \textit{and} \qquad c = -2\eta^4,$$

where $\eta \in \mathbb{Z}[i]$.

Proof. The ring of Gaussian integers $\mathbb{Z}[i]$ has the property of unique prime factorization (up to ordering), such that every Gaussian integer can be written as

$$u(1+i)^{k_0} p_1^{k_1} p_2^{k_2} \dots p_n^{k_n} = u(1+i)^{k_0} \prod_{j=1}^{n} p_j^{k_j},$$

where $u \in \{\pm 1, \pm i\}$ (i.e., u is a unit), the p_j are complex prime numbers, and $k_j \in \mathbb{N}$, for $j \in \{1, \dots, n\}$.

Let (a, c) be a zero of $F(x, z) = 2x^4 - z^3$ in Gaussian integers. Then,

$$2a^4 = c^3. \tag{15}$$

Let the prime power decompositions of a and c be

$$a = u(1+i)^{e_0} \prod_{j=1}^{l} p_j^{e_j}, \tag{16}$$

$$b = v(1+i)^{f_0} \prod_{k=1}^{m} q_k^{f_k}, \tag{17}$$

respectively, where $u, v \in \{\pm 1, \pm i\}$, and p_i, q_j are complex primes, for $e_j, f_k \in \mathbb{N}$. Replacing (16) and (17) in (15), we have

$$2u^4(1+i)^{4e_0} \prod_{j=1}^{l} p_j^{4e_j} = v^3(1+i)^{3f_0} \prod_{k=1}^{m} q_k^{3f_k}.$$

Given that $2 = -i(1+i)^2$ and $u^4 = 1$, regardless of the actual value of u, we can rewrite this as

$$-i(1+i)^{4e_0+2} \prod_{j=1}^{l} p_j^{4e_j} = v^3(1+i)^{3f_0} \prod_{k=1}^{m} q_k^{3f_k}. \tag{18}$$

By the uniqueness of the complex prime power decomposition, we have $v^3 = -i$, which implies $v = i$. For the same reason, $4e_0 + 2 = 3f_0$. We can readily see that one particular solution to this equation is $(e_0, f_0) = (1, 2)$, which gives us the general solution as $(e_0, f_0) = (3r_0 + 1, 4r_0 + 2)$, where $r_0 \geq 0$. Additionally, again from (18) and the uniqueness of the complex prime power decomposition, we must have that for every j there exists a k such that $p_j = q_k$ for $j \in \{1, \dots, l\}$ and $k \in \{1, \dots, m\}$, and $l = m$. Also, $4e_j = 3f_k$, which implies that $3|e_j$ and $4|f_k$. Thus, there exist integers r_j and s_k such that $e_j = 3r_j$ and $f_k = 4s_k$, with $r_j = s_k$.

From this and from (16) and (17), we have

$$a = u(1+i)^{3r_0+1} \prod_{j=1}^{l} p_j^{3r_j} = u(1+i)\Big((1+i)^{3r_0} \prod_{j=1}^{l} p_j^{3r_j}\Big) = u(1+i)\Big((1+i)^{r_0} \prod_{j=1}^{l} p_j^{r_j}\Big)^3,$$

and

$$c = -i(1+i)^{4r_0+2} \prod_{j=1}^{l} p_j^{4r_j} = -i(1+i)^2\Big((1+i)^{4r_0} \prod_{j=1}^{l} p_j^{4r_j}\Big) = -2\Big((1+i)^{r_0} \prod_{j=1}^{l} p_j^{r_j}\Big)^4.$$

Let $w \in \{\pm 1, \pm i\}$ be the only Gaussian integer such that $w^3 = u$ and let $\eta = w(1+i)^{r_0} \prod_{j=1}^{l} p_j^{r_j} \in \mathbb{Z}[i]$. Therefore, we have

$$a = (1+i)\eta^3 \qquad \text{and} \qquad c = -2\eta^4. \qquad \square$$

The form of zeroes to the polynomial $F(x,z) = 2x^4 - z^3$ displaying their general and imaginary parts is given in the following result.

Corollary 3.1. *Let (a, c) be a zero of the polynomial $F(x, z) = 2x^4 - z^3$ in Gaussian integers, with $a = r + si$ and $c = g + hi$. Then there exist rational integers u, v such that*

$$r = u^3 - 3u^2v - 3uv^2 + v^3, \qquad s = u^3 + 3u^2v - 3uv^2 - v^3$$

and

$$g = -2u^4 + 12u^2v^2 - 2v^4, \qquad h = -8u^3v + 8uv^3.$$

Proof. By Theorem 3.1, a solution (a, c) of the polynomial $F(x, z) = 2x^4 - z^3$ in Gaussian integers is given by

$$a = (1+i)\eta^3 \qquad \text{and} \qquad c = -2\eta^4,$$

where $\eta \in \mathbb{Z}[i]$. There exist rational integers u, v such that $\eta = u + vi$. Hence,

$$a = (1+i)(u+vi)^3 \qquad \text{and} \qquad c = -2(u+vi)^4,$$

and upon simplifying we obtain

$$\begin{aligned} a &= (u^3 - 3u^2v - 3uv^2 + v^3) + (u^3 + 3u^2v - 3uv^2 - v^3)i, \\ c &= (-2u^4 + 12u^2v^2 - 2v^4) + (-8u^3v + 8uv^3)i. \end{aligned}$$

Our assertion follows by letting

$$\begin{aligned} r &= u^3 - 3u^2v - 3uv^2 + v^3, \qquad & s &= u^3 + 3u^2v - 3uv^2 - v^3, \\ g &= -2u^4 + 12u^2v^2 - 2v^4, \qquad & h &= -8u^3v + 8uv^3. \end{aligned}$$

$\square$

4 Conclusions

In this work, we have studied the algebraic properties of the Diophantine equation $x^4 + y^4 = z^3$ in the ring of Gaussian integers, with a focus on finding the solutions of the equation when $x = y$. This leads to the determination of the zeroes of $F(x, z) = 2x^4 - z^3$. Our findings show the existence of infinitely many solutions of $F(x, z) = 0$ in the ring of Gaussian integers. In Section 2, particular solutions to $F(x, z) = 0$ are given, for which the real and imaginary components are explicitly determined. In Section 3, the general solution for $F(x, z) = 0$ is shown to be of the form $x = (1 + i)\eta^3$ and $z = -2\eta^4$, where $\eta \in \mathbb{Z}[i]$, and the general forms of the components of $x = r + si$ and $z = g + hi$ are given by $r = u^3 - 3u^2v - 3uv^2 + v^3$, $s = u^3 + 3u^2v - 3uv^2 - v^3$, $g = -2u^4 + 12u^2v^2 - 2v^4$ and $h = -8u^3v + 8uv^3$, where u, v are some rational integers.

The analytical method used in Section 2 of this paper is based on explicit analysis of the real and imaginary components of the pair of Gaussian integers $x = r + si$ and $z = g + hi$ that are zeroes of $F(x, z)$. The method examines cases and conditions under which $F_1(r, s, g, h) = 0$ and $F_2(r, s, g, h) = 0$, as we have defined them, will have trivial or non-trivial common solutions. The results obtained are used to construct the zeroes of $F(x, z)$. The algebraic techniques applied in Section 2 represent an example of study applicable to different Diophantine equations for which no general solution is known yet. It advocates examination and analysis of the real and imaginary components of the Gaussian integers satisfying such equations. It paves the way to a deeper understanding of the conditions that give rise to solutions, trivial and non-trivial, and those that lead to inconsistencies.

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